Quaternion and Euler Angles
June 9, 2019

RAHUL BHADANI rahulbhadani@email.arizona.edu

Contents

1 Euler Angles

Euler angles are three angles tuple to describe the orientation of a rigid body with respect to a fixed coordinate system. They are used to represent the orientation of a mobile frame in physics. Usually, orientation is denoted by three elemental rotation: rotation along the x-axis, y-axis, and z-axis. Typically, Euler angles are denoted as ϕ, θ and ψ, however, different authors use different notations. As a result, when talking about Euler angles, a definition must precede them. In this article, we define Euler angles as follows:

1. ϕ: Rotation of body about its fixed X-axis, also called as **roll**.
2. θ: Rotation of body about its fixed Y-axis, also called as **pitch**.
3. ψ: Rotation of body about its fixed Z-axis, also called as **yaw**.

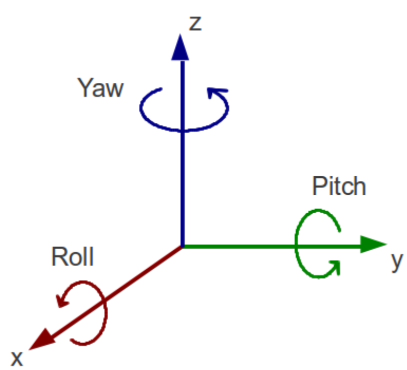

Figure 1: Euler angles.

2 Quaternions

Quaternions are another way to represent orientations and rotations of an object in three-dimensional space. Quaternions are simpler than Euler Angles, avoid ambiguity, are numerically stable and more efficient in terms of computing implementations.

2.1 Euler's rotation theorem

Any displacement of a rigid boy in three-dimensional space such that a point on the rigid body remains fixed is equivalent to a single rotation about the axis passed through that fixed point. What it means is that a combination of two or more rotation can be equivalent to a single rotation in the given scenario. The axis of rotation passing through the fixed point is called as Euler axis[1].

As we can see from Euler's rotation theorem, any rotation or a sequence of rotations of a rigid body about a fixed point is equivalent to a single rotation by a given angle θ about a fixed axis passing through the fixed point. This Euler axis is represented by $\vec{u}$. Quaternions give a simple way to represent rotation using four numbers: 3 numbers representing the axis vector and the fourth number representing an angle θ.

It is possible to convert Euler angles to Quaternions. Euler angles are easy to visualize, however, for computing purposes, Quaternions are preferred.

Let's assume that a unit quaternion is represented by

$$\mathbf{q} = \begin{bmatrix} q_w & q_x & q_y & q_z \end{bmatrix}^T \quad |\mathbf{q}|^2 = q_w^2 + q_x^2 + q_y^2 + q_z^2 = 1 \tag{1}$$

Associating a quaternion by a rotation around an axis:

$$\begin{aligned} q_w &= \cos(\alpha/2) \\ q_x &= \sin(\alpha/2)\cos(\beta_x) \\ q_y &= \sin(\alpha/2)\cos(\beta_y) \\ q_z &= \sin(\alpha/2)\cos(\beta_z) \end{aligned} \tag{2}$$

where α is a simple rotation angle in radians, $\cos(\beta_x)$, $\cos(\beta_y)$, $\cos(\beta_z)$ are direction cosines locating the axis of rotation.

3 Converting Euler Angles to Quanternion

Consider a lab axes x, y, z and rotated body axes X, Y and Z. For a situation where a body (say airplane) does rotations about yaw (Z) (say during taxing), then does rotation about pitch (Y), say during take off, and then does roll (X) say in the air - whenever. This ZYX combination is equivalent to xyz in lab axes where body rolls first, then pitches and then yaws. Do that with any solid flat body to verify. Then, the conversion is

$$\mathbf{q} = \begin{bmatrix} \cos(\psi/2) \\ 0 \\ 0 \\ \sin(\psi/2) \end{bmatrix} \begin{bmatrix} \cos(\theta/2) \\ 0 \\ \sin(\theta/2) \\ 0 \end{bmatrix} \begin{bmatrix} \cos(\phi/2) \\ \sin(\phi/2) \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

4 Quaternion to Euler

$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} atan2(2(q_w q_x + q_y q_z), 1 - 2(q_x^2 + q_y^2)) \\ asin2(2(q_w q_y - q_z q_x)) \\ atan2(2(q_w q_z + q_x q_y), 1 - 2(q_y^2 + q_z^2)) \end{bmatrix} \tag{4}$$

[1] http://math.dartmouth.edu/~euler/docs/originals/E478.pdf

5 References

1. W. G. Breckenridge, "Quaternions proposed standard conventions," NASA Jet Propulsion Laboratory, Technical Report, Oct. 1979.

2. NASA Mission Planning and Analysis Division. "Euler Angles, Quaternions, and Transformation Matrices" (PDF). NASA. Retrieved 12 January 2013.